TION

	=	concentration of carbon dioxide
C'	=	concentration of carbon monoxide
C_B	=	concentration of carbon dioxide in main gas stream
D	=	effective diffusion coefficient of carbon dioxide through porous graphite
k	=	rate constant
p	=	pressure
r	=	distance along radial direction of cylindrical sample
R	=	radius of cylindrical sample
T	=	temperature

LITERATURE CITED

1. Walker, P. L., Jr., Frank Rusinko, Jr., and L. G. Austin, "Advances in Catalysis," Vol. 11, pp. 133-221, Academic Press, New York (1959).
2. Petersen, E. E., and C. C. Wright, *Ind. Eng. Chem.*, **47**, 1624 (1955).
3. Wicke, E., "Fifth Symposium on Combustion," p. 245, Reinhold, New York (1955).
4. Prater, C. D., *Chem. Eng. Sci.*, **8**, 284 (1958).
5. Austin, L. G., Ph.D. thesis, The Pennsylvania State University, University Park, Pennsylvania (1961).
6. Gadsby, J., F. J. Long, P. Sleightholm, and K. W. Sykes, *Proc Roy. Soc. (London)*, **A193**, 357 (1948).
7. Lewis, W. K., E. R. Gilliland, and G. T. McBride, Jr., *Ind. Eng. Chem.*, **41**, 1213 (1949).
8. Wu, P. C., D.Sc. thesis, Mass. Inst. Technol., Cambridge, Massachusetts (1949).
9. Reif, A. E., *J. Phys. Chem.*, **56**, 778 (1952).

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The Laminar-Turbulent Transition for Fluids with a Yield Stress

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The pumping of semifluid substances (that is substances which flow as fluids only when stressed beyond a finite yield stress) is of considerable practical interest in fields ranging from sewage treatment to nuclear reactor design. In many of these applications it is desirable to know the conditions which determine the onset of turbulent flow.

The onset of turbulent flow can be predicted quite accurately for the flow of Newtonian fluids in ducts of various geometries (*1*), and for the nonisothermal (*2*) and isothermal (*3*) pipe flow of non-Newtonian fluids characterized by the power-law rheological model. However in reference *2* it was shown that the treatment of data obtained with semifluid substances in terms of Metzner and Reed's (*4*) point slope technique did not permit the correlation of the flow transition data for these fluids. The purpose of the present paper is to compare the author's (*1*) method of predicting the laminar-turbulent flow transition with isothermal pipe-flow data for such semifluids and to present the theoretical equations for flow in other geometries.

THEORETICAL DEVELOPMENT

Pipe Flow

A rather general criterion for the onset of turbulence has been proposed (*1*) which, for the case of flow in straight pipes of circular cross section, may be written as*

$$K = \frac{1}{2} \frac{\rho}{dp/dz} \frac{d}{dr} (v_z^2) \qquad (1)$$

If Equation (1) is evaluated at $r = \bar{r}$, the radius at which the flow field is least stable (*1*) (obtained from the condition $dK/dr = 0$), and K is set equal to the constant (*1*) value $\kappa = 404$, one can solve for $\bar{v}_c$, the critical velocity of flow. This is usually most conveniently done in dimensionless form by expressing $\bar{v}_c$ as a Reynolds number.

In order to use Equation (1) an expression for $v(r)$ is required. Such an expression can be obtained by integration of the rheological equation of the fluid together with the equations of motion. A simple rheological equation for fluids exhibiting yield stresses is the linear Bingham (*5*) equation, which for cylindrical geometry is

$$\tau_{rz} = \pm \tau_o - \eta \frac{dv_z}{dr} \qquad (2)$$

when $|\tau_{rz}| > |\tau_o|$. If $|\tau_{rz}| \leqq |\tau_o|$, the rheological equation is $dv_z/dr = 0$. In Equation (2) η is the plastic viscosity or coefficient of rigidity, and τ_o is the yield stress.

From Equation (2) and the equation of motion the following expressions for the velocity profile can be obtained:

$$v(\xi) = \frac{\tau_w r_w}{2\eta} [1 - 2\alpha(1-\xi) - \xi^2]; \quad \alpha \leqq \xi \leqq 1 \qquad (3)$$

$$v(\xi) = \frac{\tau_w r_w}{2\eta} (1-\alpha)^2; \; 0 \leqq \xi \leqq \alpha \qquad (4)$$

where $\xi = r/r_w$ is a dimensionless radius and $\alpha = \tau_o/\tau_w = r_o/r_w$ is the dimensionless radius of the unsheared plug in the central core of the flow field. From Equations (3) and (4) the average velocity can be obtained as

$$\bar{v} = \frac{\tau_w r_w}{4\eta} \left(1 - \frac{4}{3}\alpha + \frac{1}{3}\alpha^4 \right) \qquad (5)$$

which is the familiar Buckingham (*6*) equation.

From the nature of the flow field and the stability parameter (*1*), it is evident that $\alpha < \bar{\xi} < 1$, where $\bar{\xi}$ is the point of minimum stability. Therefore from Equations (1) and (3) one can obtain

$$\bar{\xi} = \alpha + (1-\alpha)\sqrt{1/3} \qquad (6)$$

From Equations (1), (3), (5), and

*The relation between Equation (1) and a similar parameter proposed by Ryan and Johnson (*3*) is discussed in reference *1*.

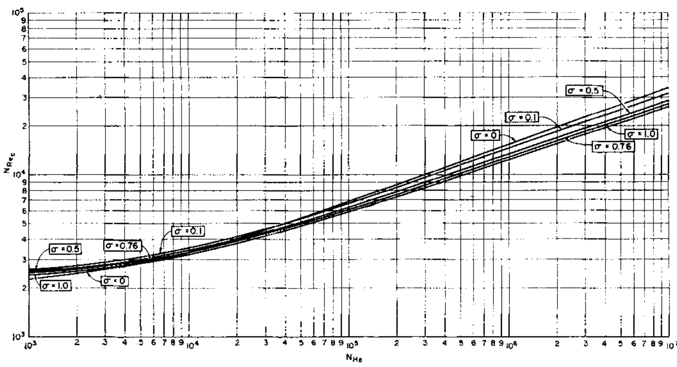

Fig. 1. Variation of N_{Re_c} with N_{He} for Bingham flow in annuli.

(6) one finds that the quantity $d\bar{v}_c\rho/\eta \equiv N_{Re_c}$ is given by

$$N_{Re_c} = \kappa\sqrt{27}\,\frac{1 - \frac{4}{3}\alpha_c + \frac{1}{3}\alpha_c^4}{(1-\alpha_c)^3} \quad (7)$$

where $\alpha_c = \tau_o/\tau_{wc}$ is the value of α corresponding to the laminar-turbulent transition, and $\kappa\sqrt{27} = 2{,}100$ (see reference *1*).

In order to use Equation (7) one needs an expression for α_c. Such an expression can be obtained by noting that, with the help of Equation (5), one can rearrange the expression for N_{Re_c} to give

$$N_{Re_c} = \frac{1 - \frac{4}{3}\alpha_c + \frac{1}{3}\alpha_c^4}{8\alpha_c} N_{He} \quad (8)$$

where $N_{He} = \rho d^2 \tau_o/\eta^2$ is the Hedstrom number (*7*) which involves only the geometry of the system and the fluid properties. From Equations (7) and (8) one finds

$$\frac{\alpha_c}{(1-\alpha_c)^3} = \frac{N_{He}}{16{,}800} \quad (9)$$

Thus N_{Re_c} can be calculated from Equations (8) and (9) and a knowledge of the physical properties of the fluid and the pipe system.

The above treatment may be extended to include flow between parallel plates and in concentric annuli.

Parallel Plates

The analogue of Equation (1) for parallel plates is $K = \frac{1}{2}\frac{\rho}{dp/dz} d(v_z^2)/dy$. By repeating the above procedure for the parallel plate geometry one obtains the following set of equations:

$$v(\phi) = \frac{3\bar{v}\,[1 - 2\alpha(1-\phi) - \phi^2]}{2\left[1 - \frac{3}{2}\alpha + \frac{1}{2}\alpha^3\right]}; \qquad \alpha \leqq \phi \leqq 1 \quad (10)$$

$$v(\phi) = \frac{3\bar{v}\,(1-\alpha)^2}{2\left(1 - \frac{3}{2}\alpha + \frac{1}{2}\alpha^3\right)}; \qquad 0 \leqq \phi \leqq \alpha \quad (11)$$

$$\bar{v} = \frac{h\tau_w}{3\eta}\left(1 - \frac{3}{2}\alpha + \frac{1}{2}\alpha^3\right) \quad (12)$$

$$\bar{\phi} = \alpha + (1-\alpha)\sqrt{1/3} \quad (13)$$

$$N_{Re_c} = \frac{1}{8}\sqrt{\frac{2}{3}}\;\frac{1 - \frac{3}{2}\alpha_c + \frac{1}{2}\alpha_c^3}{\alpha_c} N_{He} \quad (14)$$

$$\frac{\alpha_c}{(1-\alpha_c)^3} = \frac{N_{He}}{22{,}400} \quad (15)$$

where $2h$ is the separation of the plates, $\phi = y/h$ is the dimensionless distance measured from the midplane between the plates, $\tau_w = -h(dp/dz)$, and $d_p = 4h\sqrt{2/3}$, the diameter used in N_{He} and N_{Re_c}, is an equivalent diameter which has been found useful (*1*) in the Newtonian flow limit ($\alpha \to 0$).

Concentric Annuli

In dealing with the flow of Bingham fluids in concentric annuli one must pay careful attention to the sign of τ_o in Equation (2) due to the change of sign of dv_z/dr at the point of maximum velocity. A detailed treatment of the (algebraically) somewhat complicated solution of the laminar flow equations for the Bingham model has been given by Fredrickson and Bird (*8*). Since K is inherently positive (*1*), only that portion of Fredrickson and Bird's velocity profile for which $-dv_z/dr > 0$ need be considered. If their expression for this part of the velocity profile is written in terms of the radial variable $\chi = (r - r_1)/(r_2 - r_1)$, and the same procedure used above is followed, one obtains the following expression which must be solved for $\bar{\chi}$ the point of minimum stability:

$$\frac{(1-\sigma)}{f^2}\left[2\lambda(\lambda-\beta_o)\ln f + (1-\sigma)(1+\sigma-2\beta_o)(1-\bar{\chi})\right]\left[f^2 + \lambda(\lambda-\beta_o)\right] - \frac{2(1-\sigma)}{f^2}(\lambda-f)^2(\lambda+f-\beta_o)^2 = 0 \quad (16)$$

where $f = \sigma + (1-\sigma)\bar{\chi}$, $\sigma = r_1/r_2$, $\beta_o = 2\tau_o(1-\sigma)/P\delta$, $P = -dp/dz$, $\delta = r_2 - r_1$, and λ is obtained (*8*) as the solution of the following equation:

$$2\lambda(\lambda-\beta_o)\ln\left[\frac{\lambda-\beta_o}{\sigma\lambda}\right] - 1 + (\beta_o+\sigma)^2 + 2\beta_o(1-\lambda) = 0 \quad (17)$$

In terms of the equivalent diameter (*1*) $d_e = (r_2 - r_1)\sqrt{8\psi}$, where

$$\psi = \frac{(1+\sigma^2)\ln\sigma + (1-\sigma^2)}{2(1-\sigma)^2\ln\sigma} \quad (18)$$

the following result is obtained:

$$N_{Re} = \frac{1}{4}\,\frac{1 - \sigma^4 - 2\lambda(\lambda-\beta_o)(1-\sigma^2) - \frac{4}{3}\beta_o(1+\sigma^3) + \frac{1}{3}\beta_o(2\lambda-\beta_o)^3}{\beta_o(1+\sigma)(1-\sigma)^2\sqrt{8\psi}} N_{He} \quad (19)$$

In Equation (19) the parameter β_o corresponds to α_c in the previous results. The value of β_o is obtained from the Hedstrom number as follows:

$$\frac{N_{He}}{12{,}928}\,\frac{(1-\sigma)^2}{\psi} = \frac{-\beta_o f}{\left[2\lambda(\lambda-\beta_o)\ln f + (1-\sigma)(1+\sigma-2\beta_o)(1-\bar{\chi})\right](\lambda-f)(\lambda+f-\beta_o)} \quad (20)$$

In order to obtain numerical values of N_{Re_c} from the above relations Equations (16), (17), and (20) must be solved simultaneously for the values of β_o, λ, and $\bar{\chi}$. This rather tedious process has been carried out numerically for several values of the parameters involved, and the results* are

* A detailed discussion and complete tabulation of all numerical calculations are contained in the report by the author, "A Generalized Criterion for the Laminar-Turbulent Transition in the Flow of Fluids," Union Carbide Nuclear Company, Oak Ridge Gaseous Diffusion Plant, (K-1531), November 19, 1962.

shown graphically in Figure 1.

Comparison with Experiment

Unfortunately no experimental data appear to be available with which to test the calculations for flow of Bingham fluids through concentric annuli or between parallel plates. However there are available in the literature many sets of pipe-flow data obtained with a wide variety of semifluid materials. These data have been analyzed in terms of the Bingham model and plotted in Figure 2. The solid line in Figure 2 was calculated from Equations (8) and (9). There is appreciable scatter in the data, but for $N_{He} < 10^5$ the data are in good agreement with the calculated curve. For $N_{He} > 10^5$ the data tend to fall above the curve. In order to explain this trend one must consider Figure 3 in which the solid curve is a plot of α_c vs. N_{He} as calculated from Equation (9), and the data were obtained from some of the sources listed in Figure 2. It is observed in Figure 3 that for $N_{He} > 10^5$ the data points definitely fall below the calculated curve. The value of $N_{He} = 10^5$ is seen to correspond to $\alpha_c \simeq 0.55$. It is thus apparent that when $\alpha_c > 0.5$ (the radius of the unsheared plug is greater than half the pipe radius) the analysis fails to describe the physical situation accurately. This failure is very probably due to the inadequacy of the simple linear Bingham model to represent the flow profile accurately. That this is a plausible explanation can be demonstrated quantitatively by performing the analogous calculations with a more elaborate and realistic model.

PIPE FLOW SOLUTION FOR POWELL-EYRING MODEL

A rheological equation which represents a wide variety of experimental data very well (especially for Bingham fluids with large N_{He}) is the nonlinear Powell-Eyring (9) equation

$$\tau_{rz} = \mu_\infty \gamma + \frac{1}{B} \sinh^{-1} (\gamma/A) \quad (21)$$

where $\gamma = -\,dv_z/dr$, and A, B, and μ_∞ are constants.

Equation (21) more accurately represents the velocity profile than does Equation (2). In an actual fluid the sharp step-function nature of Equation (2), which requires the existence of a discontinuous shear rate distribution in the pipe, is never realized. Equation (21) permits a very slowly varying shear rate distribution in the central region of the flow field, but nevertheless one which varies continuously from zero shear rate at the axis to a maximum at the wall, without the physically unreal discontinuity of the Bingham model.

Equation (21) can be rewritten as

$$\xi T_w = \omega \Gamma + \sinh^{-1} \Gamma \quad (22)$$

where $T_w = B\tau_w$, $\Gamma = \gamma/A$, and $\omega = AB\mu_\infty$. In terms of these variables the expression for the volumetric flow rate Q can be written (9a) in the form

$$q' = \frac{1}{3}\Gamma_w - \frac{1}{3}\int_0^{\Gamma_w} \xi^3 \, d\Gamma \quad (23)$$

where $q' = q/A$ and $q = Q/\pi r_w^3$. By introducing Equation (22) into Equation (23) and performing the integration one obtains

$$q' = \frac{1}{3}\Gamma_w - \frac{1}{3T_w^3}\left\{ \frac{1}{4}\omega^3\Gamma_w^4 + \omega^2\left[\Gamma_w^3 \sinh^{-1}\Gamma_w - \Gamma_w^2\sqrt{1+\Gamma_w^2} + \frac{2}{3}(1+\Gamma_w^2)^{3/2} - \frac{2}{3} \right] + \frac{3}{4}\omega \left[(2\Gamma_w^2+1)(\sinh^{-1}\Gamma_w)^2 - 2\Gamma_w \sqrt{1+\Gamma_w^2}\sinh^{-1}\Gamma_w + \Gamma_w^2 \right] + \Gamma_w(\sinh^{-1}\Gamma_w)^3 - 3\sqrt{1+\Gamma_w^2}(\sinh^{-1}\Gamma_w)^2 + 6\left[\Gamma_w \sinh^{-1}\Gamma_w - \sqrt{1+\Gamma_w^2} + 1\right] \right\} \quad (24)$$

where $T_w = \omega\Gamma_w + \sinh^{-1}\Gamma_w$.

If the procedure described above for pipe flow is followed with Equation (22), the following equation is obtained, which must be solved for $\bar{\Gamma}$ (the value of $\bar{\Gamma}$ at the position $\bar{\xi}$ where the fluid is least stable):

$$1 + \bar{\Gamma}^2\left(2 + \frac{3}{2}\omega\sqrt{1+\bar{\Gamma}^2}\right) - \left(\frac{1}{2}\omega\Gamma^2_{wc} + \sqrt{1+\Gamma^2_{wc}}\right)\sqrt{1+\bar{\Gamma}^2} = 0 \quad (25)$$

In Equation (25) Γ_{wc} is the value of Γ_w at transition and $\bar{\Gamma}$ is related to $\bar{\xi}$ through Equation (22).

Equation (25) must be solved simultaneously with

$$\kappa = \frac{1}{8}\frac{\omega^2\bar{\Gamma}}{\Gamma^2_{wc}}\left[\frac{1}{2}\omega(\Gamma^2_{wc} - \bar{\Gamma}^2) + \sqrt{1+\Gamma^2_{wc}} - \sqrt{1+\bar{\Gamma}^2}\right] N'_{He} \quad (26)$$

where $N'_{He} = d^2\rho/\mu_\infty^2 B$ is the Powell-Eyring analogue of the Hedstrom number. Once the value of Γ_{wc} has been found, it can be used in Equation (24) to calculate q'_c from which one can then calculate the critical Reynolds number from the relation $N'_{Re_c} = \frac{1}{2}\omega q'_c \, N_{He}$ where $N'_{Re_c} = d\bar{v}_c\rho/\mu_\infty$.

One set of the data of Gregory (10) for flow of river-mud slurries through a 4-in. pipe was calculated in terms of both the Bingham model and the Powell-Eyring model. In terms of the Bingham model the ratio N_{Re_c} (exp't)/N_{Re_c} (theory) = 1.13, whereas for the Powell-Eyring model the same ratio was 1.03. Since $N_{He} = 2.8 \times 10^5$, which is in the range where deviations from the simple theory are clear, the closer agreement between the experiment and theory for the Powell-Eyring model is an indication of the inadequacy of the Bingham model to represent the flow system accurately when $N_{He} > 10^5$.

SUMMARY AND CONCLUSIONS

The range of application of the method of reference *1* for predicting the onset of turbulence [which has been successfully applied in an equivalent form to the isothermal (3) and nonisothermal (2) pipe flow of pseudoplastic fluids and to the flow of Newtonian fluids in ducts having various geometries (*1*)] has been extended to include the case of flow of fluids possessing yield stresses. The method was applied to flow in pipes, concentric

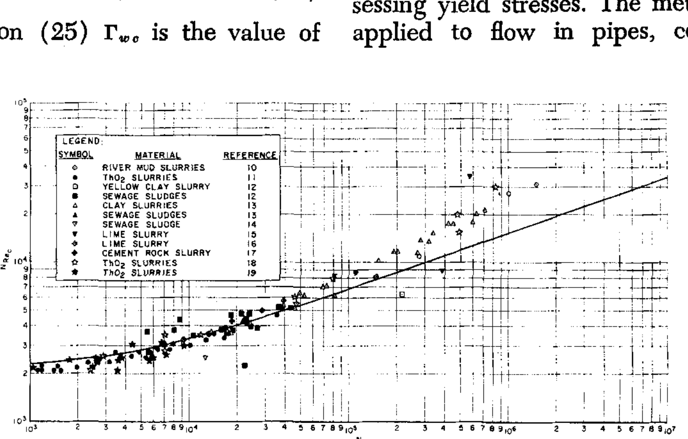

Fig. 2. Variation of N_{Re_c} with N_{He} for Bingham flow in tubes.

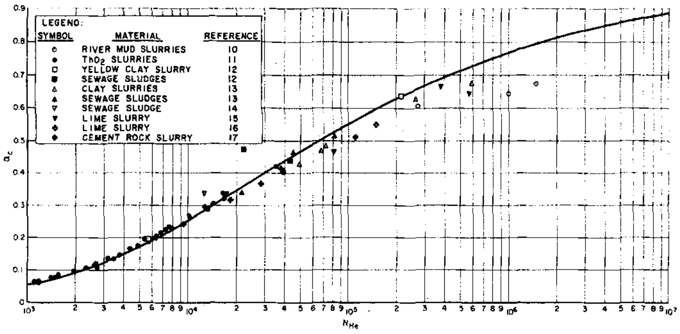

Fig. 3. Variation of α_c with N_{He} for Bingham flow in tubes.

annuli, and between parallel plates in terms of the linear Bingham plastic rheological equation (*20*). The pipe-flow calculations were compared with literature data and found to agree well for $N_{He} < 10^5$.

The systematic deviations of the data from the theoretical curve for $N_{He} > 10^5$ were explained as due to the inadequacy of the linear Bingham model to describe the flow profile accurately when the diameter of the central unsheared plug becomes roughly half that of the pipe. This explanation was examined quantitatively by use of the more complicated Powell-Eyring (9) model. Closer agreement with the mud-slurry data of Gregory (*10*) was obtained than when these data were treated in terms of the Bingham model, thus indicating the inadequacy of the simpler model at high N_{He} values. Therefore if great accuracy is desired for $N_{He} > 10^5$, one must resort to a more complex (and usually nonlinear) rheological equation in order to predict the critical Reynolds number, although the linear Bingham model will give conservative results in this range.

It has previously (2) been demonstrated that transition flow data for semifluid materials could not be correlated when these fluids were treated as generalized power-law fluids by the use of Metzner and Reed's (4) point slope technique in combination with the results of reference 3. The success of the present results therefore indicates that their method, while useful in some cases, does not correctly account for the effect of the plug on the stability properties of the flow field and hence may not be applied to this problem.

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The author wishes to thank Mr. D. M. Eisenberg of the Oak Ridge National Laboratory who made available his extensive unpublished flow data for thorium oxide slurries.

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NOTATION

A = constant in Powell-Eyring equation
B = constant in Powell-Eyring equation
d = pipe diameter
d_a = equivalent diameter for concentric annuli
d_p = equivalent diameter for parallel plates
dp/dz = axial pressure gradient
f = $\sigma + (1 - \sigma)\overline{\chi}$
h = half separation between parallel plates
K = stability parameter, defined by Equation (1)
N_{He} = Hedstrom number, $\rho\tau_o d^2/\eta^2$, (may also use d_a, d_p)
N'_{He} = Powell-Eyring Hedstrom number, $\rho d^2/\mu_\infty^2 B$
N_{Re_c} = Reynolds number, $d\overline{v}_c\rho/\eta$, (may also use d_a, d_p)
N'_{Re_c} = Powell-Eyring Reynolds number, $d\overline{v}_c\rho/\mu_\infty$
P = $-dp/dz$
q = pseudo shear rate, $Q/\pi r_w^3$
q' = q/A
Q = volumetric flow rate
r = Radial coordinate
r_1, r_2 = inner and outer radii of concentric annuli
$\overline{r}$ = radius of minimum stability
r_w = pipe radius
T_w = $B\tau_w = \omega\Gamma_w + \sinh^{-1}\Gamma_w$
v_z = axial velocity component
$\overline{v}$ = area mean velocity of flow, $Q/\pi r_w^2$
$\overline{v}_c$ = critical mean velocity at transition
y = position coordinate measured from midplane between parallel plates

Greek Letters

α = τ_o/τ_w
α_c = τ_o/τ_{wc}
β_o = $2\tau_o(1 - \sigma)/P\delta$
γ = shear rate, $-dv_z/dr$
Γ = γ/A
Γ_w = value of Γ at pipe wall
$\overline{\Gamma}$ = value of Γ at point of minimum stability
δ = separation between tubes of annulus, $r_2 - r_1$
η = plastic viscosity or coefficient of rigidity in Equation (2)
κ = critical value of stability parameter = 404
λ = parameter in Fredrickson and Bird's treatment of Bingham flow in annuli
μ_∞ = constant in Powell-Eyring equation
ξ = dimensionless radial coordinate, r/r_w
ρ = fluid density
σ = annular radius ratio, r_1/r_2
τ_o = Bingham yield stress in Equation (2)
τ_{rz} = stress component
τ_w = value of τ_{rz} at $r = r_w$
ϕ = y/h
χ = $(r - r_1)/(r_2 - r_1)$
ψ = function of σ defined by Equation (18)
ω = dimensionless grouping of Powell-Eyring constants, $AB\mu_\infty$

LITERATURE CITED

1. Hanks, R. W., *A.I.Ch.E. Journal*, **9**, No. 1, p. 45 (1963).
2. ———, and E. B. Christiansen, *ibid.*, **8**, 467 (1962).
3. Ryan, N. W., and M. M. Johnson, *ibid.*, **5**, 433 (1959).
4. Metzner, A. B., and J. C. Reed, *ibid.*, **1**, 434 (1955).
5. Bird, R. B., W. E. Stewart, and E. N. Lightfoot, "Transport Phenomena," pp. 102-103, Wiley, New York (1960).
6. Buckingham, E., *Proc. Am. Soc. Testing Materials*, **21**, 1154 (1921).
7. Hedstrom, B. O. A., *Ind. Eng. Chem.*, **44**, 651 (1952).
8. Fredrickson, A. G., and R. B. Bird, *ibid.*, **50**, 347 (1958).
9. Salt, D. L., N. W. Ryan, and E. B. Christiansen, *J. Colloid Sci.*, **6**, 146 (1951).
9a. Sisko, A. W., *Ind. Eng. Chem.*, **50**, 1789 (1958).
10. Gregory, W. B., *Mech. Eng.*, **49**, 609 (1927).
11. Eisenberg, D. M., private communication to the author.
12. Babbitt, H. E., and D. H. Caldwell, *Univ. Ill. Eng. Exp't Sta. Bull. No. 319* (Nov. 14, 1939).
13. ———, *Univ. Ill. Eng. Exp't Sta. Bull. No. 323* (Nov. 19, 1940).
14. Merkel, W., quoted in references *12* and *13*; *Beiheft zum Gesundheits-Ingenieur, Reihe II*, **Heft 14**, (1934).
15. Stevens, W. E., Ph.D. thesis, Univ. Utah, Salt Lake City, Utah (1953).
16. Alves, G. E., D. F. Boucher, and R. L. Pigford, *Chem. Eng. Progr.*, **48**, 385 (1952).
17. Wilhelm, R. H., D. M. Wroughton, and W. L. Loeffel, *Ind. Eng. Chem.*, **31**, 622 (1939).
18. Thomas, D. G., *A.I.Ch.E. Journal*, **6**, 631 (1960).
19. Nimmo, R. H., and A. S. Kitzes, *ORNL Rept. No. CF-56-12-124* (Dec. 7, 1956).
20. Prager, W., "Introduction to Mechanics of Continua," p. 136, Ginn & Co., Boston, Massachusetts (1961).

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